ENERGY CONDITION AT THE END OF A NONSTATIONARY CRACK

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The plane problem of the theory of elasticity is considered. It is assumed that in the neighborhood of the tip of an arbitrarily moving crack the stresses have a singularity of order $r^{-1/2}$. On this assumption a general expression is obtained for the distribution of the stress-tensor components in the given neighborhood. This distribution is determined by the two parameters N and P. In the case of stresses symmetrical about the line of the crack ($P = 0$) the angular distribution does not depend on the intensity coefficient N and is determined only by the velocity of the crack at the given instant and the transverse and longitudinal wave velocities. On the same assumptions it is shown that the energy condition obtained by Craggs for the particular case of steady-state motion is a necessary condition for the arbitrarily moving crack. Irwin [1] and Cherepanov [2] have studied these questions in the quasi-static approximation.

1. Singularity at the end of a moving crack. In the static case a stress singularity of order $r^{-1/2}$, where r is the distance from the crack tip, is known to exist at the end of a crack. A similar result has been obtained in a number of dynamic problems. Assuming a singularity of the same order, we seek the angular distribution of the stresses around the tip of an arbitrarily moving crack.

We describe the motion by means of the potentials Φ and Ψ, which satisfy the equation

$$c_1^2 \Delta\Phi = \partial^2\Phi / \partial t^2, \quad c_2^2 \Delta\Psi = \partial^2\Psi / \partial t^2 \tag{1.1}$$

Here, c_1 and c_2 are the longitudinal and transverse wave velocities. We find the potentials in the neighborhood of the crack tip in the following form:

$$\Phi = r^{3/2}\varphi(\vartheta, t), \quad \Psi = r^{3/2}\psi(\vartheta, t)$$
$$r = \sqrt{[x - l(t)]^2 + y^2}, \qquad \vartheta = \operatorname{arc\,tg} \frac{y}{x - l(t)} \tag{1.2}$$

Here, x, y is a coordinate system with axis directed along the crack; $l(t)$ is the crack tip coordinate. We introduce the variable $X = x - l(t)$; then

$$\frac{\partial\Phi}{\partial t} = \frac{\partial\Phi}{\partial X}(-\dot{l}) + o(r^{1/2})$$
$$\frac{\partial^2\Phi}{\partial t^2} = \frac{\partial^2\Phi}{\partial X^2} + \dot{l}^2 + o(r^{-1/2}) \tag{1.3}$$

Here, a dot denotes a derivative with respect to time. Substituting (1.2) into (1.1) and using (1.3), we obtain

$$\left(1 - \frac{\dot{l}^2}{c_1^2}\right)\frac{\partial^2\Phi}{\partial X^2} + \frac{\partial^2\Phi}{\partial y^2} = o(r^{-1/2})$$
$$\left(1 - \frac{\dot{l}^2}{c_2^2}\right)\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial^2\Psi}{\partial y^2} = o(r^{-1/2})$$

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Neglecting lower-order terms, we arrive at a system that exactly coincides with the system obtained in the problem of the steady-state propagation of a crack at constant velocity $V = l^{\circ}$ (Craggs [3]):

$$\left(1-\frac{V^2}{c_1^2}\right)\frac{\partial^2\Phi}{\partial X^2}+\frac{\partial^2\Phi}{\partial y^2}=0, \qquad \left(1-\frac{V^2}{c_2^2}\right)\frac{\partial^2\Psi}{\partial X^2}+\frac{\partial^2\Psi}{\partial y^2}=0 \tag{1.4}$$

Following Craggs, we introduce

$$z_1 = X + i\,(1 - V^2/c_1^2)^{1/2}y, \quad z_2 = X + i\,(1 - V^2/c_2^2)^{1/2}y \tag{1.5}$$

The solution of system (1.4) satisfying conditions (1.2) has the form:

$$\begin{aligned}\Phi &= \operatorname{Re}\,[a_1 z_1^{3/2} + i a_2 z_1^{3/2}]\\ \Psi &= \operatorname{Re}\,[b_1 z_2^{3/2} + i b_2 z_2^{3/2}]\end{aligned} \tag{1.6}$$

Here, a_1, a_2, b_1 and b_2 are constants. Two of these constants are found from the boundary conditions at the edges of the crack. Confining ourselves to the case of finite external forces in the region of the crack tip, we require that

$$\sigma_{yy} = \sigma_{xy} = 0 \text{ at } \vartheta = \pi \tag{1.7}$$

As the two remaining parameters we select the intensity coefficients for σ_{yy} and σ_{xy} on the axis of the crack approaching from the outside:

$$\sigma_{yy} = N/\pi\sqrt{X}, \quad \sigma_{xy} = P/\pi\sqrt{X} \qquad \text{at } \vartheta = 0 \tag{1.8}$$

To write relations (1.7) and (1.8) in terms of potentials (1.6), we used Craggs expressions for the stress-tensor components

$$\begin{aligned}\mu^{-1}\sigma_{xx} &= \left(2+\frac{V^2}{c_2^2}-2\frac{V^2}{c_1^2}\right)\frac{\partial^2\Phi}{\partial X^2}-2\left(1-\frac{V^2}{c_2^2}\right)^{1/2}\frac{\partial^2 K}{\partial X^2}\\ \mu^{-1}\sigma_{xy} &= -2\left(1-\frac{V^2}{c_1^2}\right)^{1/2}\frac{\partial^2\chi}{\partial X^2}-\left(2-\frac{V^2}{c_2^2}\right)\frac{\partial^2\Psi}{\partial X^2}\\ \mu^{-1}\sigma_{yy} &= -\left(2-\frac{V^2}{c_2^2}\right)\frac{\partial^2\Phi}{\partial X^2}+2\left(1-\frac{V^2}{c_2^2}\right)^{1/2}\frac{\partial^2 K}{\partial X^2}\end{aligned} \tag{1.9}$$

Here,

$$\begin{aligned}\frac{\partial^2\Phi}{\partial X^2}+i\frac{\partial^2 X}{\partial X^2} &= \frac{3}{4}\left(a_1 z_1^{-1/2}+i a_2 z_1^{-1/2}\right)\\ \frac{\partial^2\Psi}{\partial X^2}+i\frac{\partial^2 K}{\partial X^2} &= \frac{3}{4}\left(b_1 z_2^{-1/2}+i b_2 z_2^{-1/2}\right)\end{aligned} \tag{1.10}$$

where μ is the second Lamé constant.

These formulas are obtained by substituting into Hooke's law the displacements expressed in terms of potentials (1.6), by means of Eqs. (1.4).

Substituting (1.9), (1.10) in (1.7), (1.8) and solving the system linear in a_1, a_2, b_1, and b_2 thus obtained, we find

$$\begin{aligned}a_1 &= \frac{4}{3}\frac{N(2-V^2/c_2^2)}{\pi\mu D(V)}, & b_1 &= \frac{4}{3}\frac{P(2-V^2/c_2^2)}{\pi\mu D(V)}\\ a_2 &= -\frac{8}{3}\frac{P(1-V^2/c_2^2)^{1/2}}{\pi\mu D(V)}, & b_2 &= \frac{8}{3}\frac{N(1-V^2/c_1^2)^{1/2}}{\pi\mu D(V)}\end{aligned} \tag{1.11}$$

Here

$$D(V) = 4\,(1-V^2/c_1^2)^{1/2}(1-V^2/c_2^2)^{1/2}-(2-V^2/c_2^2)^2$$

Combining (1.9), (1.10), and (1.11), we obtain the following expressions for the stress tensor components:

$$\pi\mu D(V)\sigma_{xx} = r^{-1/2}\left\{\left(2+\frac{V^2}{c_2^2}-2\frac{V^2}{c_1^2}\right)\left(1-\frac{V^2}{c_1^2}\sin^2\vartheta\right)^{-1/4}\left[N\left(2-\frac{V^2}{c_2^2}\right)\cos\frac{\lambda_1}{2}\right.\right.$$

$$\left.\left.-2P\left(1-\frac{V^2}{c_2^2}\right)^{1/2}\sin\frac{\lambda_1}{2}\right]-2\left(1-\frac{V^2}{c_2^2}\right)^{1/2}\left(1-\frac{V^2}{c_2^2}\sin^2\vartheta\right)^{-1/4}\left[2N\left(1-\frac{V^2}{c_1^2}\right)^{1/2}\cos\frac{\lambda_2}{2}-P\left(2-\frac{V^2}{c_2^2}\right)\sin\frac{\lambda_2}{2}\right]\right\}$$

$$\pi\mu D(V)\sigma_{xy} = r^{-1/2}\left\{-2\left(1-\frac{V^2}{c_1^2}\right)^{1/2}\left(1-\frac{V^2}{c_1^2}\sin^2\vartheta\right)^{-1/4}\left[-N\left(2-\frac{V^2}{c_2^2}\right)\sin\frac{\lambda_1}{2}\right.\right.$$

$$\left.\left.-2P\left(1-\frac{V^2}{c_2^2}\right)^{1/2}\cos\frac{\lambda_1}{2}\right]-\left(2-\frac{V^2}{c_2^2}\right)\left(1-\frac{V^2}{c_2^2}\sin^2\vartheta\right)^{-1/4}\left[2N\left(1-\frac{V^2}{c_1^2}\right)^{1/2}\sin\frac{\lambda_2}{2}+P\left(2-\frac{V^2}{c_2^2}\right)\cos\frac{\lambda_2}{2}\right]\right\} \qquad (1.12)$$

$$\pi\mu D(V)\sigma_{yy} = r^{-1/2}\left\{-\left(2-\frac{V^2}{c_2^2}\right)\left(1-\frac{V^2}{c_1^2}\sin^2\vartheta\right)^{-1/4}\left[N\left(2-\frac{V^2}{c_2^2}\right)\cos\frac{\lambda_1}{2}\right.\right.$$

$$\left.\left.-2P\left(1-\frac{V^2}{c_2^2}\right)^{1/2}\sin\frac{\lambda_1}{2}\right]+2\left(1-\frac{V^2}{c_2^2}\right)^{1/2}\left(1-\frac{V^2}{c_2^2}\sin^2\vartheta\right)^{-1/4}\left[2N\left(1-\frac{V^2}{c_1^2}\right)^{1/2}\cos\frac{\lambda_2}{2}-P\left(2-\frac{V^2}{c_2^2}\right)\sin\frac{\lambda_2}{2}\right]\right\}$$

$$\lambda_n = \operatorname{arc\,tg}\,(1-V^2/c_n^2)^{1/2}\,\operatorname{tg}\nu \quad (n=1,2)$$

It should be noted that in the case of stresses symmetrical about the line of the crack (P = 0) the quantity N enters into (1.12) as a coefficient. Thus, the angular stress distribution does not depend on the intensity coefficient and has the same form for all motions of the crack.

Naturally, it coincides with the results obtained in [3-6]. The property obtained makes it possible to transfer the results of the investigations of various authors relating to particular problems to an arbitrarily moving crack with P = 0. This includes the possibility, detected by Yoffe [6], of branching of the crack to a certain critical velocity. Hence it follows that the form of the isochromes calculated by Baker [5] in his problem is common to all problems with a symmetrical stress distribution.

2. Energy condition at the end of a crack. Following Cherepanov [2], we write the energy conservation equation for the process of deformation of a perfectly elastic body with a developing crack. We assume that an energy T is required for the formation of unit length of the crack. The value of T is determined by the physical properties of the body and, generally speaking, may be a function of the parameters of motion of the crack. We isolate around the end of the crack at time t_0 a circular region G of radius ε with boundary γ. For the region of the body outside γ conservation of energy is guaranteed by the equations of motion of a perfectly elastic body. For the region G

$$2\int_0^{\varepsilon}\left(P_x\frac{\partial u_x}{\partial t}+P_y\frac{\partial u_y}{\partial t}\right)dx+\int_{\gamma}\left(X_n\frac{\partial u_x}{\partial t}+Y_n\frac{\partial u_y}{\partial t}\right)d\gamma=\frac{d}{dt}\iint_G\left\{W+\frac{\rho}{2}\left[\left(\frac{\partial u_x}{\partial t}\right)^2+\left(\frac{\partial u_y}{\partial t}\right)^2\right]\right\}dG+2\dot{l}T \qquad (2.1)$$

Here, P_x and P_y denote the vector of the external forces applied to the edges of the crack from within the crack; u_x and u_y represent the displacement vector; X_n and Y_n represent the vector of the elastic forces at the boundary; W is the elastic potential; and ρ is density.

The first term in this equation is the intensity of the external forces applied in the crack itself; the second term is the intensity of the elastic forces at the boundary of the region γ; the third term is the rate of change of elastic and kinetic energy in the region G; and the fourth term is the energy expended on crack formation per unit time.

For a sufficiently small region G it is possible to use the form of the singularity at the tip of an arbitrarily moving crack found in the preceding section; in this case the first term in (2.1) can be neglected (as the region G decreases without bound, it tends to zero).

After making all the calculations, we obtain

$$N^2\left(1-\frac{V^2}{c_1^2}\right)^{1/2}+P^2\left(1-\frac{V^2}{c_2^2}\right)^{1/2}=2\pi\mu D(V)\frac{c_2^2}{V^2} \qquad (2.2)$$

This expression is the analog of Griffith's energy condition [7] for a nonstationary moving crack. The same expression was obtained in Craggs' particular problem [3]. Instead of T we introduce the parameter K frequently used in the theory of brittle fracture:

$$K=\sqrt{2\pi\mu T/(1-\nu)} \qquad (\nu \text{ is Poisson's ratio})$$

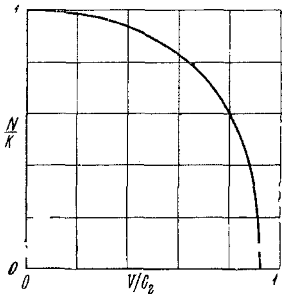

Fig. 1

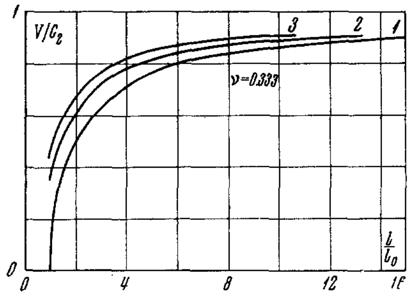

Fig. 2

The graph $N^2(V)/K^2$ for $P = 0$ and $\nu = 0.25$ is presented in Fig. 1, where c_R is the Rayleigh wave velocity. It is clear from the graph that as $V \to 0$ condition (2.2) goes over into the static condition for the equilibrium crack [8] $N = K$.

Equation (2.2) can be used for solving the following problem.

At the initial instant external loads are applied to a body with a stationary crack: it is required to find the motion of the crack.

The problem is solved in two stages:

1) We determine the parameters N and P for the given external loads and an arbitrary law of motion of a cut coinciding at the initial instant with the stationary crack.

2) By means of Eq. (2.2) from the entire set of trajectories we select that which corresponds to the motion of the crack.

Unfortunately, even for very simple regions and types of loads the solution of the first problem presents considerable difficulties. The literature on the nonsteady motion of cracks includes only the work of Broberg [4] on the self-similar propagation of a crack from zero at constant velocity and that of Baker [5] on the motion at constant velocity of a semi-infinite crack initially at rest. Broberg's result can be used for the approximate solution of the problem posed above in the case of a linear isolated crack in an infinite space. At the initial instant let a pressure $p \geq p_0$, where p_0 is the critical pressure corresponding to the equilibrium state, develop in the crack. At $t > 0$ the parameter N increases and when it becomes equal to K, the crack starts to move. At time t let it have the dimension l and velocity V. We assume that at this instant N is the same as for a Broberg crack developing from zero at velocity V to a length l under the action of the same load p:

$$N = \frac{Q\pi}{V^4} c_2^2 \left(\frac{l}{2c_1}\right)^{1/2} \frac{D(V)}{(1 - V^2/c_1^2)^{1/2}}$$

$$Q = p\frac{\beta^2(1-\beta^2)}{J(\beta)}, \qquad \beta = \frac{V}{c_1}, \qquad k = \frac{c_2}{c_1} \tag{2.3}$$

$$J(\beta) = [(1-4k^2)\beta^2 + 4k^4] K(\sqrt{1-\beta^2}) - \beta^{-2}[\beta^4 - 4k^2(1+k^2)\beta^2 + 8k^4] E(\sqrt{1-\beta^2}) -$$
$$- 4k^2(1-\beta^2) K(\sqrt{1-(\beta/k)^2}) + 8k^4\beta^{-2}(1-\beta^2) E(\sqrt{1-(\beta/k)^2})$$

where K (x), E (x) are complete elliptic integrals of the first and second kinds.

In this problem the parameter P is equal to zero.

Substituting (1.3) into (2.2), we obtain the relation $V(l)$, whose graph is presented in Fig. 2 for various values of p. Curves 1, 2, and 3 correspond to $p = p_0$, $p = 2p_0$, $p = 3p_0$. The energy of crack formation T is assumed constant. With regard to the curve $V(l)$ we know a priori that it begins at the point $(l_0, 0)$ corresponding to the initial position of the crack and tends toward the same asymptote $V = c_R$. Clearly, the greatest deviation from the exact solution will occur in the initial stage of motion, where the discrepancy between the initial data of the problem posed and the Broberg problem is greatest and where, moreover, the maximum of the acceleration dV/dt, whose effect is not taken into account in the given model, is located.

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